数理解析研究所講究録
第 1891 巻 2014 年 170-182

REMARKS ON STRICHARTZ ESTIMATES FOR SCHRÖDINGER EQUATIONS WITH POTENTIALS SUPERQUADRATIC AT INFINITY

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1. Introduction

This note is a review of author's recent work [8] which is concerned with the Strichartz estimates for variable coefficient Schrödinger equations with electromagnetic potentials growing supercritically at spatial infinity.

Consider a Schrödinger operator with variable coefficients and potentials:

$$\widetilde{P}=\frac{1}{2}(D_j-A_j(x))g^{jk}(x)(D_k-A_k(x))+V(x),\ D_j:=-i\partial/\partial x_j,\ x\in\mathbb{R}^d.$$

with the standard summation convention. We impose the following.

Assumption A.

- $g^{jk},A_j,V\in C^\infty(\mathbb{R}^d;\mathbb{R})$.
- $(g^{jk}(x))_{j,k}$ is symmetric and uniformly elliptic:

$$g^{jk}(x)\xi_j\xi_k\geq c|\xi|^2$$

on $\mathbb{R}^{2d}$ with some positive constant $c>0$.
- There exists $m\geq 2$ such that, for any $\alpha\in\mathbb{Z}_+^d:=\mathbb{N}^d\cup\{0\}$,

$$|\partial_x^\alpha g^{jk}(x)|+\langle x\rangle^{-m/2}|\partial_x^\alpha A_j(x)|+\langle x\rangle^{-m}|\partial_x^\alpha V(x)|\leq C_\alpha\langle x\rangle^{-|\alpha|}. \tag{1.1}$$

- $\widetilde{P}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.

Remark 1.1. If we assume in addition to the first three conditions as above that $V\geq -C\langle x\rangle^2$ with some constant $C>0$, then $\widetilde{P}$ is essentially self-adjoint. It is also known that this condition is almost optimal for the essential self-adjointness of $\widetilde{P}$. However, $\widetilde{P}$ can be essentially self-adjoint even if $V\leq -C\langle x\rangle^k$ with $k>2$ if strongly divergent magnetic fields are present near infinity. More precisely, we set

$$|B(x)|=\Big(\sum_{j,k=1}^d|B_{jk}(x)|^2\Big)^{1/2},B_{jk}=\partial_jA_k-\partial_kA_j.$$

Note that $|B(x)|\lesssim\langle x\rangle^{m/2-1}$ under the above assumption. Then, Iwatsuka [4] proved that If $V(x)+|B(x)|\gtrsim-\langle x\rangle^2$ then $\widetilde{P}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.

Let us denote by P the self-adjoint extension of $\widetilde{P}$ on $L^2(\mathbb{R}^d)$. Then we consider the time-dependent Schrödinger equation

$$i\partial_t u=Pu,\ t\in\mathbb{R};\quad u|_{t=0}=u_0\in L^2(\mathbb{R}^d). \tag{1.2}$$

The solution is given by $u(t)=e^{-itP}u_0$ by Stone's theorem, where e^{-itP} denotes a unitary propagator on $L^2(\mathbb{R}^d)$ generated by P.

In this paper we are interested in the (local-in-time) *Strichartz estimates* of the forms:

$$||e^{-itP}u_0||_{L_T^pL^q}\leq C_T||\langle H\rangle^\gamma u_0||_{L^2}, \tag{1.3}$$

where $\gamma \geq 0$, $L^p_T L^q := L^p([-T,T]; L^q(\mathbb{R}^d))$ and (p,q) satisfies the *admissible condition*

$$2 \leq p, q \leq \infty, \quad 2/p = d(1/2 + 1/q), \quad (d,p,q) \neq (2,2,\infty). \tag{1.4}$$

Strichartz estimates can be regarded as L^p-type smoothing properties of Schrödinger equations and have been widely used in the study of nonlinear Schrödinger equations (see, *e.g.*, [2]).

If P satisfies Assumption A with $m < 2$, the nontrapping condition (see below) and the following long-range condition:

$$|\partial_x^\alpha (g^{jk}(x) - \delta_{jk})| \leq C_\alpha \langle x \rangle^{-\mu-|\alpha|}, \quad \mu > 0,$$

then it has been shown in [6, 7] that $e^{-itP}u_0$ satisfies (1.3) with $\gamma = 0$ which is the same as in the free case at least locally in time.

When $m > 2$ the situation becomes considerably different. More precisely, if $g^{jk} = \delta_{jk}$ and $A \equiv 0$, then the following has been proved by Yajima-Zhang [13]:

Theorem 1.2 (Theorem 1.3 of [13]). *Let $H = -\Delta/2 + V$ satisfy Assumption A and*

$$V(x) \geq C\langle x \rangle^m \quad \textit{for } |x| \geq R, \tag{1.5}$$

with some $R, C > 0$. Then, for any $\varepsilon, T > 0$ and (p,q) satisfying (1.4),

$$||e^{-itH}u_0||_{L^p_T L^q} \leq C_{T,\varepsilon} ||\langle H \rangle^{\frac{1}{p}(\frac{1}{2}-\frac{1}{m})+\varepsilon} u_0||_{L^2}. \tag{1.6}$$

The aim of this note is to extend their result to the variable coefficient case. Moreover, we will remove the additional ε-loss in the flat case (*i.e.*, $g^{jk} \equiv \delta_{jk}$).

To state our main results, we here introduce some notations on the classical system. Let $k(x,\xi) = \frac{1}{2}g^{jl}(x)\xi_j\xi_l$ be the classical kinetic energy function and $(y_0(t,x,\xi), \eta_0(t,x,\xi))$ the Hamilton equation generated by k:

$$\dot{y}_0(t) = \nabla_\xi k(y_0(t), \eta_0(t)), \ \dot{\eta}_0(t) = -\nabla_x k(y_0(t), \eta_0(t))$$

with the initial condition $(y_0, \eta_0)|_{t=0} = (x, \xi)$. Note that the Hamiltonian vector field $H_k = \nabla_\xi k \cdot \nabla_x - \nabla_x k \cdot \nabla_\xi$ is complete on $\mathbb{R}^{2d}$ and $(y_0(t), \eta_0(t))$ thus exists for all $t \in \mathbb{R}$.

Assumption B.

- Nontrapping condition: For any $(x,\xi) \in \mathbb{R}^{2d}$ with $\xi \neq 0$,

$$|y_0(t,x,\xi)| \to +\infty \quad \text{as } t \to \pm\infty.$$

- Convexity near infinity: There exists $f \in C^\infty(\mathbb{R}^d)$ satisfying

$$f \geq 1, \quad \lim_{|x|\to+\infty} f(x) = +\infty, \quad \partial_x^\alpha f \in L^\infty(\mathbb{R}^d) \text{ for any } |\alpha| \geq 2$$

  and constants $c, R > 0$ such that

$$\{k, \{k, f\}\}(x,\xi) \geq ck(x,\xi) \quad \text{on} \quad \{(x,\xi) \in \mathbb{R}^{2d};\ f(x) \geq R\},$$

  where $\{k, f\} = H_k f$ is the Poisson bracket.

Remark 1.3. It is easy to see that if $\sup_{|\alpha|\leq 2} \langle x \rangle^{|\alpha|} |\partial_x^\alpha (g^{jk}(x) - \delta_{jk})|$ is sufficiently small, then $\partial_t^2(|y_0(t)|^2) \gtrsim |\xi|^2$ and hence Assumption B holds with $f(x) = 1 + |x|^2$. For more examples satisfying Assumption B, we refer to [3, Section 2].

We now state main results.

Theorem 1.4. *Let $d \geq 2$ and P satisfy Assumptions A and B. Then, for any $T, \varepsilon > 0$ and (p,q) satisfying* (1.4), *there exists $C_{T,\varepsilon} > 0$ such that*

$$||e^{-itP}u_0||_{L^p_T L^q} \leq C_{T,\varepsilon}\big(||\langle D\rangle^{\frac{1}{p}(1-\frac{2}{m})+\varepsilon}u_0||_{L^2} + ||\langle x\rangle^{\frac{1}{p}(\frac{m}{2}-1)+\varepsilon}u_0||_{L^2}\big). \tag{1.7}$$

For the flat case, we can remove the additional ε-loss as follows.

Theorem 1.5. *Let $d \geq 3$ and $H = \frac{1}{2}(D - A(x))^2 + V(x)$ satisfy Assumption A. Then, for any $T > 0$ and (p,q) satisfying* (1.4) *there exists $C_T > 0$ such that*

$$||e^{-itH}u_0||_{L^p_T L^q} \leq C_T\big(||\langle D\rangle^{\frac{1}{p}(1-\frac{2}{m})}u_0||_{L^2} + ||\langle x\rangle^{\frac{1}{p}(\frac{m}{2}-1)}u_0||_{L^2}\big). \tag{1.8}$$

Remark 1.6. Suppose that V satisfies (1.5). Then we can assume $P \geq 1$ without loss of generality and P hence is uniformly elliptic in the sense that $p(x,\xi) \approx |\xi|^2 + \langle x\rangle^m$, where

$$p(x,\xi) = \frac{1}{2}g^{jk}(x)(\xi_j - A_j(x))(\xi_k - A_k(x)) + V(x).$$

By the standard parametrix construction for P, we see that, for any $1 < q < \infty$ and $s \geq 0$

$$||P^{s/2}v||_{L^q} + ||v||_{L^q} \approx ||\langle D\rangle^s v||_{L^q} + ||\langle x\rangle^{ms/2}v||_{L^q}.$$

(see, *e.g.*, [13, Lemma 2.4]). The right hand side of (1.7) (resp. (1.8)) is thus dominated by $||\langle P\rangle^{(1/2-1/m)/p+\varepsilon}u_0||_{L^2}$ (resp. $||\langle H\rangle^{(1/2-1/m)/p}u_0||_{L^2}$). Therefore, our result is a generalization and improvement of Theorem 1.2.

Remark 1.7. The additional ε-loss in (1.7) is only due to the use of the smoothing effect:

$$||\langle x\rangle^{-1/2-\varepsilon}E_{1/m}e^{-itP}u_0||_{L^2_T L^2} \leq C_{T,\varepsilon}||u_0||_{L^2}, \quad \varepsilon > 0,$$

where E_s is a pseudodifferential operator with the symbol $(k(x,\xi) + \langle x\rangle^m)^{s/2}$. It is well known that this estimate does not holds when $\varepsilon = 0$ even for $P = -\frac{1}{2}\Delta + \langle x\rangle^m$ (see [9]).

1.1. **Notations.** We write $L^q = L^q(\mathbb{R}^d)$ if there is no confusion. $W^{s,q} = W^{s,q}(\mathbb{R}^d)$ is the Sobolev space with the norm $||f||_{W^{s,q}} = ||\langle D\rangle^s f||_{L^q}$. For Banach spaces X and Y, $||\cdot||_{X\to Y}$ denotes the operator norm from X to Y. For constants $A, B \geq 0$, $A \lesssim B$ means that there exists some universal constant $C > 0$ such that $A \leq CB$. $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. We always use the letter P (resp. H) to denote variable coefficient (resp. flat) Schrödinger operators. For $h \in (0,1]$, we set

$$p^h(x,\xi) = \frac{1}{2}g^{jk}(x)(\xi_j - hA_j(x))(\xi_k - hA_k(x)) + h^2V(x).$$

2. Preliminaries

In this section we record some known results on the semiclassical pseudodifferential calculus and the Littlewood-Paley theory. This section also discuss local smoothing effects for the propagator e^{-itP} under Assumption B.

First of all we collect basic properties of the semiclassical pseudodifferential operator (h-ΨDO for short). We omit proofs and refer to [10] for the details. Set a metric on the phase space $T^*\mathbb{R}^d \cong \mathbb{R}^{2d}$ defined by $g = dx^2/\langle x\rangle^2 + d\xi^2/\langle\xi\rangle^2$. For a g-continuous weight function $m(x,\xi)$, we use Hörmander's symbol class $S(m,g)$, which is the space of smooth functions on $\mathbb{R}^{2d}$ satisfying $|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}m(x,\xi)\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}$. To a symbol $a \in C^\infty(\mathbb{R}^{2d})$ and $h \in (0,1]$, we associate the h-ΨDO $a(x,hD)$ defined by

$$a(x,hD)f(x) = (2\pi h)^{-d}\int e^{i(x-y)\cdot\xi/h}a(x,\xi)f(y)dyd\xi, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class. For a h-ΨDO A, we denote its symbol by $\mathrm{Sym}(A)$, *i.e.*, $A = a(x, hD)$ if $a = \mathrm{Sym}(A)$. It is known as the Calderón-Vaillancourt theorem that for any symbol $a \in C^\infty(\mathbb{R}^{2d})$ satisfying $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}$, $a(x, hD)$ is extended to a bounded operator on $L^2(\mathbb{R}^d)$ with a uniform bound in $h \in (0,1]$. Moreover, if $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{-\gamma}$ with some $\gamma > d$, then $a(x,hD)$ is extended to a bounded operator from L^q to L^r with bounds

$$||a(x,hD)||_{L^q \to L^r} \le C_{qr} h^{-d(1/q-1/r)}, \quad 1 \le q \le r \le \infty, \tag{2.1}$$

where $C_{qr} > 0$ is independent of $h \in (0,1]$. These bounds follow from the Schur lemma and the Riez-Thorin interpolation theorem (see, *e.g.*, [1, Proposition 2.4]). For two symbols $a \in S(m_1, g)$ and $b \in S(m_2, g)$, $a(x,hD)b(x,hD)$ is also a h-ΨDO with the symbol $a\sharp b(x,\xi) = e^{ihD_\eta D_z} a(x,\eta) b(z,\xi)|_{z=x,\eta=\xi} \in S(m_1 m_2, g)$, which has the expansion

$$a\sharp b - \sum_{|\alpha|<N} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a \cdot \partial_x^\alpha b \in S(h^N \langle x\rangle^{-N} \langle\xi\rangle^{-N} m_1 m_2, g). \tag{2.2}$$

In particular, we have $\mathrm{Sym}([a(x,hD), b(x,hD)]) - \frac{h}{i}\{a,b\} \in S(h^2\langle x\rangle^{-2}\langle\xi\rangle^{-2}, g)$, where $\{a,b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b$ is the Poisson bracket. The symbol of the adjoint $a(x,hD)^*$ is given by $a^*(x,\xi) = e^{ihD_\eta D_z} a(z,\eta)|_{z=x,\eta=\xi} \in S(m_1, g)$ which has the expansion

$$a^* - \sum_{|\alpha|<N} \frac{h^{|\alpha|}}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha a \in S(h^N \langle x\rangle^{-N} \langle\xi\rangle^{-N} m_1, g). \tag{2.3}$$

We also often use the following which is a direct consequence of (2.2):

Lemma 2.1. *Let $a \in S(m_1, g)$ and $b \in S(m_2, g)$. If $b \equiv 1$ on $\operatorname{supp} a$, then for any $N \ge 0$,*

$$a(x,hD) = a(x,hD)b(x,hD) + h^N r_N(x,hD) = b(x,hD)a(x,hD) + h^N \widetilde{r}_N(x,hD)$$

with some $r_N, \widetilde{r}_N \in S(\langle x\rangle^{-N}\langle\xi\rangle^{-N} m_1 m_2, g)$.

2.1. **Littlewood-Paley estimates.** We here prove Littlewood-Paley estimates, which will be used to reduce the proof of the estimates (1.7) to that of energy localized Strichartz estimates. Here and in what follows, the summation over h, $\sum_h$, means that h takes all negative powers of 2 as values, *i.e.*, $\sum_h := \sum_{h=2^{-j}, j\ge 0}$.

Proposition 2.2. *For $h \in (0,1]$, there exist two symbols Ψ_0^h and Ψ_1^h such that the following statements are satisfied with constants independent of h:*
(1) (*Symbol estimates*) *$\{\Psi_k^h\}_{h\in(0,1]}$ are bounded in $S(1, h^{4/m}dx^2 + d\xi^2/\langle\xi\rangle^2)$, i.e.,*

$$|\partial_x^\alpha \partial_\xi^\beta \Psi_k^h(x,\xi)| \le C_{\alpha\beta} h^{(2/m)|\alpha|} \langle\xi\rangle^{-|\beta|}, \quad k = 0,1.$$

(2 (*Support property*)

$$\operatorname{supp} \Psi_0^h \subset \{(x,\xi);\ h^2\langle x\rangle^m \lesssim 1,\ |\xi|^2 \approx 1\}, \tag{2.4}$$

$$\operatorname{supp} \Psi_1^h \subset \{(x,\xi);\ h^2\langle x\rangle^m \approx 1,\ |\xi|^2 \lesssim 1\}. \tag{2.5}$$

(3) (*Littlewood-Paley estimates*) *For any $q \in [2,\infty)$,*

$$||v||_{L^q} \lesssim ||v||_{L^2} + \sum_{k=0,1} \Big(\sum_h ||\Psi_k^h(x,hD)v||_{L^q}^2 \Big)^{1/2}. \tag{2.6}$$

In order to prove Proposition 2.2, we prepare two lemmas. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp}\varphi \subset [-1,1]$, $\varphi \equiv 1$ on $[-1/2,1/2]$ and $0 \le \varphi \le 1$. We set

$$\psi_0(x,\xi) = \varphi\Big(\frac{\langle x\rangle^{m/2}}{\varepsilon|\xi|}\Big),\ \psi_1 = 1 - \psi_0,$$

where $\varepsilon > 0$ is a sufficiently small constant such that $p(x,\xi) \approx |\xi|^2$ if $\langle x\rangle^m \le \varepsilon|\xi|^2$. It is easy to see that $\operatorname{supp}\psi_0 \subset \{(x,\xi); \langle x\rangle^m \le \varepsilon^2|\xi|^2\}$, $\operatorname{supp}\psi_1(\varepsilon) \subset \{(x,\xi); \langle x\rangle^m \ge \varepsilon^2|\xi|^2/2\}$ and that $\psi_0, \psi_1 \in S(1,g)$ for each $\varepsilon > 0$.

Lemma 2.3. *For any $\theta \in C_0^\infty(\mathbb{R}^d)$ supported away from the origin and any $N > d$, there exists a bounded family $\{\Psi_0^h\}_{h\in(0,1]} \subset S(1, h^{4/m}dx^2 + d\xi^2/\langle\xi\rangle^2)$ satisfying (2.4) such that*

$$\|\theta(hD)\psi_0(x,D) - \Psi_0^h(x,hD)\|_{L^2\to L^q} \le C_{qN}h^{N-d(1/2-1/q)}, \quad h \in (0,1],\ q \in [2,\infty)$$

Moreover, if we set

$$\Psi_1^h(x,\xi) := \theta(h^{m/2}x)\psi_1(x,\xi/h),$$

then $\{\Psi_1^h\}_{h\in(0,1]}$ is bounded in $S(1, h^{4/m}dx^2 + d\xi^2/\langle\xi\rangle^2)$ and satisfies (2.5).

Proof. Choose $\widetilde\theta \in C_0^\infty(\mathbb{R}^d)$ so that $\widetilde\theta$ is supported away from the origin and that $\widetilde\theta \equiv 1$ on $\operatorname{supp}\theta$. Then we learn by (2.2) (with $h = 1$) that

$$\theta(hD)\psi_0(x,D) = \theta(hD)\widetilde\theta(hD)\psi_0(x,D) = \theta(hD)\widetilde\psi_0^h(x,D) + \theta(hD)\widetilde r_N^h(x,D),$$

where $\widetilde\psi_0^h \in S(1,g)$ and $\widetilde r_N^h \in S(\langle x\rangle^{-N}\langle\xi\rangle^{-N}, g)$. Since $|\xi| \approx h^{-1}$ on $\operatorname{supp}\theta(h\xi)$, we have

$$\|\theta(hD)\widetilde r_N^h(x,D)\|_{L^2\to L^q} \le \|\theta(hD)\langle D\rangle^{-N}\|_{L^2\to L^q}\|\langle D\rangle^N\widetilde r_N^h(x,D)\|_{L^2\to L^2} \lesssim h^{N-d(1/2-1/q)}.$$

For the main term, we see that $\operatorname{supp}\widetilde\psi_0^h(\cdot,\cdot/h) \subset \{(x,\xi);\ h^2\langle x\rangle^m \lesssim 1,\ |\xi| \approx 1\}$ and that $\{\widetilde\psi_0^h(\cdot,\cdot/h)\}_{h\in(0,1]}$ is bounded in $S(1,g)$. In particular, $\widetilde\psi_0^h(x,D)$ can be regarded as a h-ΨDO with the symbol $\widetilde\psi_0^h(\cdot,\cdot/h)$. (2.2) again implies that there exist bounded families $\{\Psi_0^h\}_{h\in(0,1]} \subset S(1,g)$ and $\{r_N^h\}_{h\in(0,1]} \subset S(\langle x\rangle^{-N}\langle\xi\rangle^{-N}, g)$ such that

$$\theta(hD)\widetilde\psi_0^h(x,D) = \Psi_0^h(x,hD) + h^N r_N^h(x,hD).$$

It is easy to see that Ψ_0^h obeys the desired properties.

On the other hand, since $\operatorname{supp}\partial_x^\alpha\partial_\xi^\beta\psi_1 \subset \operatorname{supp}\psi_0$ for any $|\alpha+\beta| \ge 1$, we learn $|\xi| \approx h^2\langle x\rangle^m \approx 1$ on $\operatorname{supp}\theta(h^{2/m}x) \cap \operatorname{supp}\partial_x^\alpha\partial_\xi^\beta\psi_1(x,\xi/h)$ as long as $|\alpha+\beta| \ge 1$. Hence $\{\Psi_1^h\}_{h\in(0,1]}$ is also bounded in $S(1, h^{4/m}dx^2 + d\xi^2/\langle\xi\rangle^2)$ and satisfies (2.5). □

Lemma 2.4. *Let $c > 1$ and consider a c-adic partition of unity:*

$$\theta_0, \theta \in C_0^\infty(\mathbb{R}^d),\ \operatorname{supp}\theta \subset \{1/c < |x| < c\},\ 0 \le \theta_0, \theta \le 1,\ \theta_0(x) + \sum_{l\ge0}\theta(c^{-l}x) = 1.$$

Then, for any $2 \le q < \infty$,

$$\|v\|_{L^q} \lesssim \|v\|_{L^2} + \Big(\sum_l \|\theta(c^{-l}D)v\|_{L^q}^2\Big)^{1/2}, \tag{2.7}$$

$$\|v\|_{L^q} \lesssim \|\theta_0(x)v\|_{L^q} + \Big(\sum_l \|\theta(c^{-l}x)v\|_{L^q}^2\Big)^{1/2}. \tag{2.8}$$

Proof. We refer to [11] for the details of the proof. □

Proof of Proposition 2.2. Set $h = 2^{-l}$. We plug $\psi_0(x,D)v$ into (2.7) with $c = 2$. By virtue of Lemma 2.3, the contribution of the error term $\theta(hD)\widetilde{r}_N^h(x,D) + h^N r_N^h(x,hD)$ is dominated by $||v||_{L^2}$ provided that $N > d(1/2 - 1/q)$. We hence have

$$||\psi_0(x,D)v||_{L^q} \lesssim ||v||_{L^2} + \Big(\sum_h ||\Psi_0^h(x,hD)v||_{L^q}^2\Big)^{1/2}.$$

The proof of the estimate for $\psi_1(x,D)v$ is similar □

2.2. **Local smoothing effects.** We here prove the local smoothing effects for e^{-itP}. Set

$$e_s(x,\xi) := (k_A(x,\xi) + \langle x\rangle^m + L(s))^{s/2},\ s \in \mathbb{R},$$

where $k_A(x,\xi) = \frac{1}{2}g^{jk}(x)(\xi_j - A_j(x))(\xi_k - A_k(x))$ and $L(s)$ is a constant depending on s. Then, $e_s \in S(e_s, dx^2/\langle x\rangle^2 + d\xi^2/e_1^2)$, that is

$$|\partial_x^\alpha \partial_\xi^\beta e_s(x,\xi)| \le C_{\alpha\beta}\ e_{s-|\beta|}(x,\xi)\langle x\rangle^{-|\alpha|}. \tag{2.9}$$

Let $E_s = e_s(x,D)$ and $\mathcal{B}^s := \{f;\ \langle x\rangle^s f \in L^2,\ \langle D\rangle^2 f \in L^2\}$. Then, for any $s \in \mathbb{R}$, there exists $L(s) > 0$ such that E_s is a homeomorphism from $\mathcal{B}^{r+s}$ to $\mathcal{B}^r$ for all $r \in \mathbb{R}$, and E_s^{-1} is also a ΨDO with the symbol $\widetilde{e}_{-s}$ in $S(e_{-s}, dx^2/\langle x\rangle^2 + d\xi^2/e_1^2)$ (see, [3, Lemma 4.1]).

We first show the energy estimates.

Lemma 2.5. *For any $s \in \mathbb{R}$ there exists $C_s > 0$ such that*

$$||E_s e^{-itP}u_0||_{L^2} \le e^{C_s|t|}||E_s u_0||_{L^2},\ t \in \mathbb{R}.$$

Proof. Set $B_s = [E_s, P]E_s^{-1}$. Then, (2.9) and the symbolic calculus show that, for any $s \in \mathbb{R}$, $B_s - B_s^*$ is bounded on L^2. Set $v(t) = E_s e^{-itP}u_0$ and compute

$$\begin{aligned}\frac{d}{dt}||v(t)||_{L^2}^2 &= \langle -i(P+B_s)v(t), v(t)\rangle + \langle v(t), -i(P+B_s)v(t)\rangle\\ &= -i\langle (B_s - B_s^*)v(t), v(t)\rangle\\ &\le C_s||v(t)||_{L^2}\end{aligned}$$

The assertion then follows from Gronwall's inequality. □

We now state the local smoothing effects for the propagator e^{-itP}.

Proposition 2.6. *Assume Assumptions A and B. Then, for any $T > 0$, $\nu > 0$ and $s \in \mathbb{R}$, there exists $C_{T,\nu,s} > 0$ such that*

$$||\langle x\rangle^{-1/2-\nu}E_{s+1/m}e^{-itP}u_0||_{L_T^2L^2} \le C_{T,\nu,s}||E_s u_0||_{L^2}. \tag{2.10}$$

Proof. By time reversal invariance, we may replace the time interval $[-T,T]$ by $[0,T]$ without loss of generality. Robbiano-Zuliy [9] proved the case when $s = 0$ only. However, by virtue of Lemma 2.5, general cases can be verified by an essentially same argument. We hence omit details. □

Remark 2.7. Assumption B is only needed for Proposition 2.6.

3. Parametrix construction

Write $\Gamma^h(L) := \{(x,\xi);\ |\xi|^2 + h^2\langle x\rangle^m < L\}$, where $L \ge 1$ is a large constant such that $\operatorname{supp}\Psi_k^h \subset \Gamma^h(L)$, $k = 0,1$. This section is devoted to construct the parametrices of propagators, localized in this energy shell, in terms of the semiclassical Fourier integral operator (h-FIO for short).

Let us first consider the solution to the Hamilton system:

$$\dot{X}_j = \frac{\partial p^h}{\partial \xi_j}(X,\Xi),\quad \dot{\Xi}_j = -\frac{\partial p^h}{\partial x_j}(X,\Xi);\quad (X(0,x,\xi),\Xi(0,x,\xi)) = (x,\xi) \in \Gamma^h(L).$$

The flow is well-defined for $|t| \le \delta h^{-2/m}$ and $(x,\xi) \in \Gamma^h(L)$ with sufficiently small $\delta > 0$. More precisely, we have an a priori bound:

$$|\Xi(t,x,\xi)|^2 + h^2\langle X(t,x,\xi)\rangle^m \le C,\quad (t,x,\xi) \in [-\delta_0 h^{-2/m}, \delta_0 h^{-2/m}] \times \Gamma^h(L).$$

Using this bound, we further obtain more precise behavior of the flow (see [8] for the detail of the proof).

Lemma 3.1 (General case). *Set $\Omega^h(R,L) := \{|x| > R\} \cap \Gamma^h(L)$. For sufficiently small $0 < \delta < \delta_0$, the following statements are satisfied:*
(1) *For any $h \in (0,1]$, $1 \le R \le h^{-2/m}$, $(t,x,\xi) \in [-\delta R, \delta R] \times \Omega^h(R,L)$,*

$$|X(t) - x| + \langle x\rangle|\Xi(t) - \xi| \le C|t|, \tag{3.1}$$

$$|\partial_x^\alpha \partial_\xi^\beta (X(t) - x)| + \langle x\rangle|\partial_x^\alpha \partial_\xi^\beta (\Xi(t) - \xi)| \le C_{\alpha\beta}\langle x\rangle^{-1}|t|,\quad |\alpha+\beta| \ge 1, \tag{3.2}$$

where constants $C, C_{\alpha\beta} > 0$ may be taken uniformly in h, R and t.
(2) *If $(Y(t,x,\xi),\xi)$ denotes the inverse map of $\Lambda(t)$, then bounds* (3.1) *and* (3.2) *still hold with $X(t)$ replaced by $Y(t)$ for $(t,x,\xi) \in [-\delta R, \delta R] \times \Omega^h(R,L)$.*
(3) *The same conclusions also hold with $R = 1$ and with $\Omega^h(R,L)$ replaced by $\Gamma^h(L)$, i.e., $X(t)$ and $Y(t)$ satisfy* (3.1) *and* (3.2) *uniformly in $h \in (0,1]$ and $(t,x,\xi) \in [-\delta,\delta] \times \Gamma^h(L)$.*

Lemma 3.2 (Flat case). *Assume that $g^{jk} \equiv \delta_{jk}$. Then, for sufficiently small $0 < \delta < \delta_0$, the followings hold uniformly with respect to $h \in (0,1]$:*
(1) *For any $(t,x,\xi) \in [-\delta h^{-2/m}, \delta h^{-2/m}] \times \Gamma^h(L)$, we have*

$$|X(t) - x| + h^{-2/m}|\Xi(t) - \xi| \le C|t| \tag{3.3}$$

$$|\partial_x^\alpha \partial_\xi^\beta (X(t) - x)| \le C_{\alpha\beta} h^{2/m}|t|,\quad |\alpha+\beta| \ge 1, \tag{3.4}$$

$$|\partial_x \Xi(t)| \le C_\alpha h^{2/m}\langle x\rangle^{-1}|t|,\ |\partial_\xi(\Xi(t) - \xi)| \le C_\alpha h^{4/m}|t|,$$

$$|\partial_x^\alpha \partial_\xi^\beta (\Xi(t) - \xi)| \le C_{\alpha\beta} h^{2/m}\langle x\rangle^{-1}|t|,\quad |\alpha+\beta| \ge 2.$$

(2) *Denote by $(Y(t,x,\xi),\xi)$ the inverse map of $\Lambda(t)$. Then the bounds* (3.3) *and* (3.4) *still hold with $X(t)$ replaced by $Y(t)$.*

We next turn into the construction of parametrices. We begin with the general case.

Theorem 3.3. *There exists $\delta > 0$ such that, for any $h \in (0,1]$ and $1 \le R \le h^{-2/m}$, the following statements are satisfied with constants independent of h and R:*
(1) *There exists a solution $S^h \in C^\infty((-\delta R, \delta R) \times \mathbb{R}^{2d})$ to the Hamilton-Jacobi equation:*

$$\begin{cases} \partial_t S^h(t,x,\xi) + p^h(x, \partial_x S^h(t,x,\xi)) = 0, & (t,x,\xi) \in (-\delta R, \delta R) \times \Omega^h(R/3, 3L), \\ S^h(0,x,\xi) = x\cdot\xi, & (x,\xi) \in \Omega^h(R/3, 3L), \end{cases} \tag{3.5}$$

such that

$$|\partial_x^\alpha \partial_\xi^\beta \left(S^h(t,x,\xi) - x\cdot\xi + tp^h(x,\xi)\right)| \le C_{\alpha\beta}\langle x\rangle^{-1-\min(|\alpha|,1)}|t|^2, \tag{3.6}$$

uniformly in $(t,x,\xi) \in (-\delta R, \delta R) \times \mathbb{R}^{2d}$.

(2) *For any* $\chi^h \in S(1,g)$ *supported in* $\Omega^h(R,L)$ *and integer* $N \ge 0$, *there exists a bounded family* $\{a^h(t); |t| \le \delta R,\ h \in (0,1]\} \subset S(1,g)$ *with* $\operatorname{supp} a^h(t) \subset \Omega^h(R/2, 2L)$ *such that*

$$e^{-itP^h/h}\chi^h(x,hD) = J_{S^h}(a^h) + Q^h(t,N),$$

where $P^h = h^2 P$ *and* $J_{S^h}(a^h)$ *is the h-FIO defined by*

$$J_{S^h}(a^h)f(x) = (2\pi h)^{-d}\int e^{i\left(S^h(t,x,\xi)-y\cdot\xi\right)/h} a^h(t,x,\xi) f(y)dyd\xi,$$

and the remainder $Q^h(t,N)$ *satisfies*

$$\sup_{|t|\le\delta R} ||Q^h(t,N)||_{L^2\to L^2} \le C_N h^{N-1-2/m}. \tag{3.7}$$

Furthermore, if $K^h(t,x,\xi)$ *denotes the kernel of* $J_{S^h}(a^h)$ *then*

$$|K^h(t,x,y)| \lesssim \min\{h^{-d},\ |th|^{-d/2}\}, \quad x,\xi\in\mathbb{R}^d,\ h\in(0,1],\ |t|\le\delta R. \tag{3.8}$$

Proof. **Construction of the phase** S^h: Define S^h on $(-\delta R, \delta R) \times \Omega^h(R/4, 4L)$ by

$$S^h(t,x,\xi) := x\cdot\xi + \int_0^t L^h(X(s,Y(t,x,\xi),\xi), \Xi(s,Y(t,x,\xi),\xi)ds,$$

where $L^h = \xi\cdot\partial_\xi p^h - p^h$ is the Lagrangian associated to p^h. A direct computation yields that S^h solves (3.5) and satisfies $(\partial_\xi S^h, \partial_x \widetilde{S}^h) = (Y(t,x,\xi), \Xi(t,Y(t,x,\xi),\xi))$. Furthermore, the conservation law, $p^h(x, \partial_x S^h(t,x,\xi)) = p^h(Y(t,x,\xi),\xi)$, holds. By virtue of Lemma 3.1 (2), taking $\delta > 0$ smaller if necessary we see that

$$h^2\langle Y(t,x,\xi)\rangle^m \le 5L, \quad (t,x,\xi)\in(-\delta R,\delta R)\times\Omega^h(R/4,4L)$$

and hence

$$|p^h(x,\partial_x S^h) - p^h| \lesssim |Y(t)-x|\int_0^1 |(\partial_x p^h)(\lambda x + (1-\lambda)Y(t),\xi)|d\lambda \lesssim \langle x\rangle^{-1}|t|.$$

The estimates for derivatives can be proved by an induction. Integrating with respect to t and using Hamilton-Jacobi equation (3.5), we see that S^h satisfies (3.6) on $\Omega^h(R/4,4L)$. We finally extend S^h to the whole space $\mathbb{R}^{2d}$ such that $S^h = x\cdot\xi - tp^h$ on $\Omega^h(R/3,3L)$.

Construction of the amplitude a^h: Let us make the following ansatz:

$$v(t,x) = \frac{1}{(2\pi h)^d}\int e^{i(S^h(t,x,\xi)-y\cdot\xi)/h} a^h(t,x,\xi)f(y)dyd\xi,$$

where $a^h = \sum_{j=0}^{N-1} h^j a_j^h$. In order to approximately solve the Schrödinger equation

$$(hD_t + P^h)v(t) = O(h^N); \quad v|_{t=0} = \chi^h(x,hD)u_0,$$

the amplitude should satisfy the following transport equations:

$$\begin{cases} \partial_t a_0^h + \mathcal{X}\cdot\partial_x a_0 + \mathcal{Y}a_0^h = 0; & a_0^h|_{t=0} = \chi^h, \\ \partial_t a_j^h + \mathcal{X}\cdot\partial_x a_j + \mathcal{Y}a_j^h + iKa_{j-1}^h = 0; & a_j^h|_{t=0} = 0, \quad 1\le j\le N-1, \end{cases} \tag{3.9}$$

where $K = -\frac{1}{2}\partial_j g^{jk}(x)\partial_k$, a vector field $\mathcal{X}$ and a function $\mathcal{Y}$ are defined by

$$\mathcal{X}(t,x,\xi) := (\partial_\xi p^h)(x,\partial_x S^h(t,x,\xi)),\ \mathcal{Y}(t,x,\xi) := [k(x,\partial_x)S^h + p_1^h(x,\partial_x S^h)](t,x,\xi).$$

The system (3.9) can be solved by the standard method of characteristics along the flow generated by $\mathfrak{X}(t,x,\xi)$. More precisely, let us consider the following ODE

$$\partial_t z(t,s,x,\xi) = \mathfrak{X}(t, z(t,s,x,\xi),\xi); \quad z(s,s) = x.$$

Then, there exists $\delta > 0$ such that, for any fixed $h \in (0,1]$, $1 \le R \le h^{-2/m}$, $z(t,s,x,\xi)$ is well-defined for $t,s \in (-\delta R, \delta R)$ and $(x,\xi) \in \Omega(R/3,3L)$, and satisfies

$$|z(t,s)-x| \le C|t-s|, \quad |\partial_x^\alpha \partial_\xi^\beta (z(t,s)-x)| \le C_{\alpha\beta}\langle x\rangle^{-1}|t-s|, \quad |\alpha+\beta| \ge 1. \quad (3.10)$$

We then define a_j, $j=0,1,...,N-1$, inductively by

$$a_0(t,x,\xi) = \chi^h(z(0,t,x,\xi),\xi)\exp\left(\int_0^t \mathcal{Y}(s,z(s,t,x,\xi),\xi)ds\right),$$

$$a_j(t,x,\xi) = -\int_0^t (iKa_{j-1})(s,z(s,t,x,\xi),\xi)\exp\left(\int_u^t \mathcal{Y}(u,z(u,t,x,\xi),\xi)du\right)ds.$$

It is easy to see from (3.10) and the support property $\operatorname{supp}\chi^h \subset \Omega^h(R,L)$ that $\operatorname{supp} a_j \subset \Omega^h(R/2,2L)$ for all $|t| \le \delta R$. Furthermore, taking $\delta > 0$ smaller if necessary we see that a_j are smooth on $\Omega(5R/12,12L/5)$. Since $\Omega^h(R/2,2L) \Subset \Omega(5R/12,12L/5) \Subset \Omega(R/3,3L)$, if we extend a_j to the whole space $\mathbb{R}^{2d}$ so that $a_j \equiv 0$ outside $\Omega^h(R/2,2L)$, then a_j are still smooth. We further learn that $a_j \in S(1,g)$ uniformly with respect to $|t| \le \delta R$ and $h \in (0,1]$. Finally, one can check by a direct computation that a_j solve the system (3.9).

Justification of the parametrix and dispersive estimates: (3.6) implies $|\partial_\xi \otimes \partial_x S^h(t,x,\xi) - \mathrm{Id}| < 1/2$ for $(t,x,\xi) \in (-\delta R,\delta R) \times \Omega^h(R/3,3L)$. Therefore, for any amplitude $b^h \in S(1,g)$ supported in $\Omega^h(R/2,2L)$,

$$\sup_{|t|\le\delta R} ||J_{S^h}(b^h)||_{L^2\to L^2} \lesssim 1, \quad h \in (0,1], \ 1 \le R \le h^{-2/m}.$$

Assume $t \ge 0$ without loss of generality. By the Duhamel formula, we have

$$e^{-itP^h/h}\chi^h(x,hD) = J_{S^h}(a^h) - \frac{i}{h}\int_0^t e^{-i(t-s)P^h/h}(hD_t + P^h)J_{S^h}(a^h)|_{t=s}ds.$$

By (3.5), (3.9) and direct computations, we obtain

$$(hD_t + P^h)J_{S^h}(a^h) = -ih^N J_{S^h}(Ka^h_{N-1}).$$

Since $\operatorname{supp} Ka^h_{N-1} \subset \Omega(R/2,2L)$ and $Ka^h_{N-1} \in S(1,g)$, $J_{S^h}(P^h a^h_{N-1})$ is bounded on L^2 uniformly in $h \in (0,1]$, $1 \le R \le h^{-2/m}$ and $0 \le t \le \delta R$, and (3.7) follows. The dispersive estimate is verified by the stationary phase method. □

Remark 3.4. It can be verified by the same argument and Lemma 3.1 (3) that for any symbol $\chi^h \in S(1,g)$ supported in $\Gamma^h(L)$, $e^{-itP^h/h}\chi^h(x,hD)$ can be approximated by a time-dependent h-FIO as above if $|t| < \delta$, and in particular obeys the dispersive estimate

$$||e^{-itP^h/h}\chi^h(x,hD)||_{L^1\to L^\infty} \lesssim \min\{h^{-d}, |th|^{-d/2}\}, \quad |t| < \delta, \ h \in (0,1].$$

We next state the flat case.

Theorem 3.5 (Flat case). *Suppose that $g^{jk} \equiv \delta_{jk}$ and $L \ge 1$. Then, there exists $\delta > 0$ such that the following statements are satisfied with constants independent of $h \in (0,1]$:*
(1) *There exists $S^h \in C^\infty((-\delta h^{-2/m}, \delta h^{-2/m}) \times \mathbb{R}^{2d})$ such that*

$$\begin{cases} \partial_t S^h(t,x,\xi) + p^h(x,\partial_x S^h(t,x,\xi)) = 0, \ (t,x,\xi) \in (-\delta h^{-2/m}, \delta h^{-2/m}) \times \Gamma^h(3L), \\ S^h(0,x,\xi) = x\cdot\xi, \ (x,\xi) \in \Gamma^h(3L), \end{cases}$$

and that

$$|\partial_x^\alpha \partial_\xi^\beta \left(S^h(t,x,\xi) - x\cdot\xi + tp^h(x,\xi)\right)| \le C_{\alpha\beta} h^{(2/m)(1+\min\{|\alpha|,1\})}|t|^2.$$

(2) *For any* $\chi^h \in S(1,g)$ *with* $\operatorname{supp}\chi^h \subset \Gamma^h(L)$ *and integer* $N \ge 0$, *there exists* $\{a^h(t); t \in (-\delta h^{-2/m}, \delta h^{-2/m}), h \in (0,1]\} \subset S(1,g)$ *with* $\operatorname{supp} a^h(t) \subset \Gamma^h(2L)$ *such that*

$$\sup_{|t|\le \delta h^{-2/m}} ||e^{-itH^h/h}\chi^h(x,hD) - J_{S^h}(a^h)||_{L^2\to L^2} \le C_N h^{N-1-2/m},$$

where the kernel of $J_{S^h}(a^h)$ *satisfies* (3.8) *for* $|t| \le \delta h^{-2/m}$.

The proof is analogous to the general case and the only difference is to use Lemma 3.2 instead of Lemma 3.1.

4. Proof of main theorems

In this section we prove Theorems 1.4 and 1.5. For simplicity, we only consider the case $d \ge 3$. The following which is a direct consequence of Theorem 3.3 and Remark 3.4.

Theorem 4.1. (1) *For any symbol* $\chi_R^h \in S(1,g)$ *supported in* $\{|x| > R\} \cap \Gamma^h(L)$,

$$||\chi_R^h(x,hD)e^{-itP}\chi_R^h(x,hD)^*||_{L^1\to L^\infty} \le C_\delta |t|^{-d/2}, \quad 0 < |t| < \delta hR,$$

(2) *For any symbol* $\chi^h \in S(1,g)$ *supported in* $\Gamma^h(L)$,

$$||\chi^h(x,hD)e^{-itP}\chi^h(x,hD)^*||_{L^1\to L^\infty} \le C_\delta |t|^{-d/2}, \quad 0 < |t| < \delta h.$$

Using Theorem 4.1, Keel-Tao's abstract theorem (see [5]) and the Duhamel formula, one can obtain the following semiclassical Strichartz estimates with an inhomogeneous term. The proof is same as that of [7, Proposition 7.4] (see also [1, Section 5]).

Proposition 4.2. *Let* $2^* = 2d/(d-2)$. *Under conditions in Theorem 4.1, we have*

$$\begin{aligned}
||\chi_R^h(x,hD)e^{-itP}u_0||_{L_T^2L^{2^*}} &\lesssim h||u_0||_{L^2} + ||\chi_R^h(x,hD)u_0||_{L^2}\\
&\quad + (hR)^{-1/2}||\chi_R^h(x,hD)e^{-itP}u_0||_{L_T^2L^2}\\
&\quad + (hR)^{1/2}||[H,\chi_R^h(x,hD)]e^{-itP}u_0||_{L_T^2L^2},\\
||\chi^h(x,hD)e^{-itP}u_0||_{L_T^2L^{2^*}} &\lesssim h||u_0||_{L^2} + ||\chi^h(x,hD)u_0||_{L^2}\\
&\quad + h^{-1/2}||\chi^h(x,hD)e^{-itP}u_0||_{L_T^2L^2}\\
&\quad + h^{1/2}||[H,\chi^h(x,hD)]e^{-itP}u_0||_{L_T^2L^2},
\end{aligned}$$

uniformly with respect to $h \in (0,1]$ *and* $1 \le R \le h^{-2/m}$.

Proof of Theorem 1.4. First of all, Proposition 2.2 and Minkowski's inequality show

$$||e^{-itP}u_0||_{L_T^2L^{2^*}} \lesssim ||u_0||_{L^2} + \sum_{k=0,1}\Big(\sum_h ||\Psi_k^h(x,hD)e^{-itP}u_0||^2_{L_T^2L^{2^*}}\Big)^{1/2},$$

with $\Psi_k^h \in S(1, h^{4/m}dx^2 + d\xi^2/\langle\xi\rangle^2)$ satisfying $\operatorname{supp}\Psi_0^h \subset \{\langle x\rangle \lesssim h^{-2/m},\ |\xi| \approx 1\}$ and $\operatorname{supp}\Psi_1^h \subset \{\langle x\rangle \approx h^{-2/m},\ |\xi| \lesssim 1\}$.

We first study $\Psi_1^h(x,hD)e^{-itP}$. The expansion formula (2.2) shows

$$\operatorname{supp}\operatorname{Sym}([P,\Psi_1^h(x,hD)]) \subset \operatorname{supp}\Psi_1^h, \quad \operatorname{Sym}([P,\Psi_1^h(x,hD)]) \in S(h^{-1+2/m},g).$$

Therefore, using Lemma 2.1 we have

$$||[P,\Psi_1^h(x,hD)]e^{-itP}u_0||_{L_T^2L^2} \lesssim h^{-1/2+1/m}||\widetilde{\Psi}_1^h(x,hD)e^{-itP}u_0||_{L_T^2L^2} + h||u_0||_{L^2}, \quad (4.1)$$

where $\widetilde{\Psi}_1^h \in S(1,g)$ is of the form $\widetilde{\Psi}_1^h(x,\xi) = \widetilde{\theta}(h^{2/m}x)\widetilde{\psi}_1(x,\xi/h)$ with $\widetilde{\theta} \in C_0^\infty(\mathbb{R}^d)$ supported in $\{|x| \approx 1\}$ and with $\widetilde{\psi}_1 \in S(1,g)$ supported in $\{|\xi|^2 \lesssim \langle x\rangle^m\}$. In particular, $\widetilde{\Psi}_1^h \equiv 1$ on $\operatorname{supp}\Psi_1^h$. Applying Proposition 4.2 to $\Psi_1^h(x,hD)e^{-itP}$ with $R \approx h^{-2/m}$ and using (4.1), we then obtain

$$\begin{aligned}&||\Psi_1^h(x,hD)e^{-itP}u_0||_{L_T^2L^{2^*}}\\ &\lesssim h||u_0||_{L^2} + ||\theta(h^{2/m}x)\psi_1(x,D)u_0||_{L^2} + ||\widetilde{\theta}(h^{2/m}x)\langle x\rangle^{m/4-1/2}\widetilde{\psi}_1(x,D)e^{-itP}u_0||_{L_T^2L^2},\end{aligned}$$

where, in the last line, we have used the fact that $h^{-1/2+1/m} \approx \langle x\rangle^{m/4-1/2}$ on $\operatorname{supp}\widetilde{\Psi}_1^h$. Combining this estimate with the following the norm equivalence

$$||v||_{L^2}^2 \approx \sum_h ||\theta(h^{2/m}x)v||_{L^2}^2 \approx \sum_h ||\widetilde{\theta}(h^{2/m}x)v||_{L^2}^2,$$

we have

$$\sum_h ||\Psi_1^h(x,hD)e^{-itP}u_0||_{L_T^2L^{2^*}}^2 \lesssim ||u_0||_{L^2}^2 + ||\langle x\rangle^{m/4-1/2}\widetilde{\psi}_1(x,D)e^{-itP}u_0||_{L_T^2L^2}^2.$$

Since $\langle x\rangle^{m\nu/2} \le e_\nu(x,\xi)$ for any $\nu \ge 0$ we conclude

$$\sum_h ||\Psi_1^h(x,hD)e^{-itP}u_0||_{L_T^2L^{2^*}}^2 \lesssim ||E_{1/2-1/m}u_0||_{L^2}^2. \quad (4.2)$$

Next we study $\Psi_0^h(x,hD)e^{-itP}u_0$. Choose a dyadic partition of unity:

$$\varphi_{-1}(x) + \sum_{0\le j\le j_h} \varphi(2^{-j}x) = 1, \quad x \in \pi_x(\operatorname{supp}\Psi_0^h),$$

where $j_h \lesssim (2/m)\log(1/h)$ and $\varphi_{-1},\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\varphi_{-1} \subset \{|x| < 1\}$ and $\operatorname{supp}\varphi \subset \{1/2 < |x| < 2\}$. We set $\varphi_j(x) = \varphi(2^{-j}x)$ for $j \ge 0$. Since $p,q \ge 2$, it follows from Minkowski's inequality that

$$||\Psi_0^h(x,hD)e^{-itP}u_0||_{L_T^2L^{2^*}}^2 \le \sum_{-1\le j\le j_h} ||\varphi_j(x)\Psi_0^h(x,hD)e^{-itP}u_0||_{L_T^2L^{2^*}}^2.$$

We here take cut-off functions $\widetilde{\varphi}_{-1},\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^d)$ and $\widetilde{\Psi}_0^h \in S(1,g)$ supported in a small neighborhood of $\operatorname{supp}\varphi_{-1}$, $\operatorname{supp}\varphi$ and $\operatorname{supp}\Psi_0^h$, respectively, so that $\widetilde{\varphi}_{-1} \equiv 1$ on $\operatorname{supp}\varphi_{-1}$, $\widetilde{\varphi} \equiv 1$ on $\operatorname{supp}\varphi$ and $\widetilde{\Psi}_0^h \equiv 1$ on $\operatorname{supp}\Psi_0^h$. Set $\widetilde{\varphi}_j(x) = \widetilde{\varphi}(2^{-j}x)$ for $j \ge 0$. Then,

$$\operatorname{supp}\widetilde{\varphi}_j\widetilde{\Psi}_0^h \subset \{|x| \approx 2^j,\ |\xi| \approx 1\}, \quad \widetilde{\varphi}_j\widetilde{\Psi}_0^h \equiv 1 \text{ on } \operatorname{supp}\varphi_j\Psi_0^h.$$

Since the symbolic calculus shows $\operatorname{supp}\operatorname{Sym}([P,\varphi_j(x)\Psi_0^h(x,hD)]) \subset \operatorname{supp}(\varphi_j\Psi_0^h)$ and

$$\operatorname{Sym}([P,\varphi_j(x)\Psi_0^h(x,hD)]) \in S(2^{-j}h^{-1},g),$$

we learn by Proposition 4.2 with $R = 2^j$ that

$$\begin{aligned}&||\varphi_j(x)\Psi_0^h(x,hD)e^{-itP}u_0||_{L_T^2L^{2^*}}\\ &\lesssim h||u_0||_{L^2} + ||\varphi_j(x)\Psi_0^h(x,hD)u_0||_{L^2} + ||\widetilde{\varphi}_j(x)\langle x\rangle^{-1/2}\langle D\rangle^{1/2}\widetilde{\Psi}_0^h(x,hD)e^{-itP}u_0||_{L_T^2L^2}.\end{aligned}$$

The almost orthogonality of φ_j and $\widetilde{\varphi}_j$ then yields

$$\sum_{-1\le j\le j_h} ||\varphi_j(x)\Psi_0^h(x,hD)e^{-itP}u_0||^2_{L^2_TL^{2^*}}$$
$$\lesssim h||u_0||^2_{L^2} + ||\Psi_0^h(x,hD)u_0||^2_{L^2} + ||\langle x\rangle^{-1/2}\langle D\rangle^{1/2}\widetilde{\Psi}_0^h(x,hD)e^{-itP}u_0||_{L^2_TL^2}.$$

We further obtain by the symbolic calculus that

$$||\langle x\rangle^{-1/2}\langle D\rangle^{1/2}\widetilde{\Psi}_0^h(x,hD)e^{-itP}u_0||_{L^2_TL^2} \lesssim ||\widetilde{\Psi}_0^h(x,hD)\langle x\rangle^{-1/2}E_{1/2}e^{-itP}u_0||_{L^2_TL^2} + h^{\frac{1}{2}}||u_0||_{L^2}.$$

Now we choose a smooth cut-off function $\widetilde{\theta} \in C_0^\infty(\mathbb{R}^d)$ supported away from the origin such that $\widetilde{\theta} \equiv 1$ on $\pi_\xi(\operatorname{supp}\Psi_0^h)$. Lemma 2.1 then yields

$$||\Psi_0^h(x,hD)(1-\widetilde{\theta}(hD))||_{L^2\to L^q} + ||\widetilde{\Psi}_0^h(x,hD)(1-\widetilde{\theta}(hD))||_{L^2\to L^q} \le Ch$$

for $2 \le q \le \infty$ and $h \in (0,1]$. We hence may replace $\Psi_0^h(x,hD)$ and $\widetilde{\Psi}_0^h(x,hD)$ by $\Psi_0^h(x,hD)\widetilde{\theta}(hD)$ and $\widetilde{\Psi}_0^h(x,hD)\widetilde{\theta}(hD)$, respectively. Then the L^2-boundedness of $\widetilde{\Psi}_0^h(x,hD)$ and the almost orthogonality of $\widetilde{\theta}(h\xi)$ imply

$$\sum_h \left(h||u_0||^2_{L^2} + ||\Psi_0^h(x,hD)\widetilde{\theta}(hD)u_0||^2_{L^2}\right) \lesssim ||u_0||^2_{L^2},$$
$$\sum_h ||\widetilde{\Psi}_0^h(x,hD)\widetilde{\theta}(hD)\langle x\rangle^{-1/2}E_{1/2}e^{-itP}u_0||^2_{L^2_TL^2} \lesssim ||\langle x\rangle^{-1/2}E_{1/2}e^{-itP}u_0||^2_{L^2_TL^2}.$$

Furthermore, we have

$$||\langle x\rangle^{-1/2}E_{1/2}e^{-itP}u_0||_{L^2_TL^2} \lesssim ||\langle x\rangle^{-1/2-m\nu/2}E_{1/2+\nu}e^{-itP}u_0||_{L^2_TL^2}.$$

We now apply Proposition 2.6 with $s = 1/2 - 1/m + \nu$ to obtain

$$\sum_h ||\Psi_0^h(x,hD)e^{-itP}u_0||^2_{L^2_TL^{2^*}} \le C_{T,\nu}||E_{1/2-1/m+\nu}u_0||^2_{L^2}, \quad T>0, \tag{4.3}$$

provided that $\nu > 0$.

Summering the estimates (4.2) and (4.3) we conclude that

$$||e^{-itP}u_0||_{L^2_TL^{2^*}} \le C_{T,\nu}||E_{1/2-1/m+\nu}u_0||_{L^2}$$
$$\le C_{T,\nu}||\langle D\rangle^{1/2-1/m+\nu}u_0||_{L^2} + C_{T,\nu}||\langle x\rangle^{m/4-1/2+\nu}u_0||_{L^2}$$

for any admissible pair (p,q) with $q<\infty$ and $\nu>0$. Finally, Theorem 1.4 can be verified by interpolation with the trivial $L^\infty_TL^2$-estimate. We refer to *e.g.*, [12] for the interpolation in weighted spaces. □

Next we prove Theorem 1.5. Hence, in what follows (in this section), we suppose that $H = \frac{1}{2}(D-A(x))^2 + V(x)$ satisfies Assumption A. In this case, we first obtain a slightly long-time dispersive estimate which is better than Theorem 4.1 (2).

Theorem 4.3. *Let $I \Subset (0,\infty)$ be an interval and $\delta > 0$ small enough. Then, for any $h \in (0,1]$ and symbol $\chi^h \in S(1,g)$ supported in $\Gamma^h(L)$,*

$$||\chi^h(x,hD)e^{-itH}\chi^h(x,hD)^*||_{L^1\to L^\infty} \le C_\delta|t|^{-d/2}, \quad 0<|t|<\delta h^{1-2/m}.$$

We also learn by this theorem and the TT^*-argument that

Proposition 4.4. *Under conditions in Theorem 4.3, we have*

$$\begin{aligned}||\chi^h(x,hD)e^{-itH}u_0||_{L^2_TL^{2^*}} \lesssim h||u_0||_{L^2} + ||\chi^h(x,hD)u_0||_{L^2} \\ + h^{-1/2+1/m}||\chi^h(x,hD)e^{-itH}u_0||_{L^2_TL^2} \\ + h^{1/2-1/m}||[H,\chi^h(x,hD)]e^{-itH}u_0||_{L^2_TL^2},\ h\in(0,1].\end{aligned}$$

Proof of Theorem 1.5. The proof is analogous to that of Theorem 1.4. The only difference compared to the previous one is the following fact:

$$\mathrm{Sym}([H,\Psi_0^h(x,hD)]) = h^{-2}\,\mathrm{Sym}([H^h,\Psi_0^h(x,hD)]) \in S(h^{-1+2/m},g), \tag{4.4}$$

which can be verified by the symbolic calculus. By Proposition 4.4 and (4.4), we have

$$||\Psi_0^h(x,hD)e^{-itH}u_0||_{L^2_TL^{2^*}} \lesssim ||\Psi_0^h(x,hD)u_0||_{L^2} + ||\widetilde{\Psi}_0^h(x,hD)E_{1/2-1/m}e^{-itH}u_0||_{L^2_TL^2}.$$

By Lemma 2.5, we then conclude

$$\sum_h ||\Psi_0^h(x,hD)e^{-itH}u_0||^2_{L^2_TL^{2^*}} \lesssim ||E_{1/2-1/m}u_0||^2_{L^2}$$

which, together with the estimates (4.2) and Lemma 2.2, implies

$$||e^{-itH}u_0||_{L^2_TL^{2^*}} \le C_T||E_{1/2-1/m}u_0||_{L^2}.$$

□

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